Book of Abstract

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Invited talks

A graceful tale

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A *graceful labeling* of a graph G with e edges is an injection from the vertex set of G to the set $\{0, 1, \ldots, e\}$ with the property that $\{|f(u) - f(v)| : uv \in E(G)\} = \{1, \ldots, e\}$. A graph G having a graceful labeling is also said a *graceful graph*. In Rosa's seminal paper [3] such a labelings are called β-valuations, the term 'graceful' was introduced by Golomb [1] .

Rosa introduced graceful labelings for tackling Ringel's conjecture [2] according to which the complete graph of order $2e+1$ can be decomposed into $2e+1$ subgraphs that are all isomorphic to a given tree with e edges. Ringel's conjecture is one of the oldest and best known open conjectures on graph decompositions and has always attracted attention.

In this talk we will report on classical and new results on graceful graphs. This is a joint work with Andrea Vietri.

References

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Blocking sets in finite projective geometries

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The aim of this talk is to give a gentle introduction to the theory of blocking sets in finite geometries. We will keep the focus on constructive issues, since there are quite some open problems, old and new, where examples are missing. I do not assume that most of the audience has a deep experience with finite geometries, thus I will try to outline the details in a way that empowers every interested participant to work on the problems introduced in the talk. We will need some background on linear algebra, finite fields and polynomials.

A *blocking set* in a finite projective plane is a set of points $\mathcal{B}$ such that $\mathcal{B}$ intersects every line of the plane in at least one point. (In hypergraph theory, such objects are called transversals or vertex covers.) These objects have been subject of serious studies since the middle of the last century. Clearly, if we add some points to a blocking set, we again get a blocking set; thus we mostly focus on *minimal* blocking sets (that is, blocking sets that do not contain a smaller blocking set). Furthermore, we will concentrate on small (and minimal) examples. There are deep structural results about minimal and small blocking sets and, also, there are several examples of such objects.

Let us consider the following generalization: a *t-fold blocking set* is a set of points which intersects every line in at least t points. Similarly as in the case of (1-fold) blocking sets, deep structural results have been known about these since a while; a main problem is that we do not know in general whether they exist or not, as there are no constructions for them! I became quite fascinated (and somewhat frustrated) when this fact became apparent to me. We will discuss some (partially) successfull attempts to construct small 2-fold (a.k.a. *double*) blocking sets.

Note that a double blocking set $\mathcal{B}$ has the property that every line is spanned by its intersection with $\mathcal{B}$. If we generalize this to higher dimensional projective geometries, we arrive at the notion of *strong blocking sets*: a point set $\mathcal{B}$ is a strong blocking set in an N-dimensional projective space if every hyperplane is spanned by its intersection with $\mathcal{B}$. (It follows that $\mathcal{B}$ is an N-fold blocking set with respect to hyperplanes.) In the past few years, strong blocking sets have been studied intensively because of their connection with so-called minimal codes. We will discuss constructions of (small) strong blocking sets.

Graph decompositions

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A graph G can be decomposed into $H_1, \ldots, H_t$ if the edge set of G can be partitioned into copies of $H_1, \ldots, H_t$. This talk will be about decomposing complete graphs into trees (i.e. when G is complete and $H_1, \ldots, H_t$ are trees). Here there has been a lot of progress in recent years (such as the solution of a conjecture of Ringel for large trees), and also a lot of things we still don't know (such as a conjecture of Gyárfás). The focus of this talk will not be about presenting any new results, but rather surveying existing ones and presenting simplified proofs of some of them.

Transversal Embedding

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A classical question in graph theory is to find sufficient conditions which guarantee that a graph G contains a given spanning subgraph H. A colourful variant of this problem has graphs $G_1, \ldots, G_s$ on the same vertex set, where $s \geq e(H)$ and we think of each graph as having a different colour, and the goal is to find a *transversal* (or *rainbow*) copy of H that contains at most one edge from each graph G_i.

This is a new area with a lot of recent progress as well as many open problems.

I hope to sketch the proofs of three results in this area, each of which takes a different approach:

- A rainbow version of Mantel's theorem by Aharoni, DeVos, de la Maza, Montejano and Šámal, which proves a best possible condition on the number of edges required in graphs G_1, G_2, G_3 to guarantee a rainbow triangle.
- A rainbow version of Dirac's theorem by Joos and Kim, which proves a best possible condition on the minimum degree required in n-vertex graphs $G_1, \ldots, G_n$ to guarantee a rainbow Hamilton cycle.
- A rainbow version of the Hajnal-Szemerédi theorem by Montgomery, Müyesser and Pehova, which proves an asymptotically best possible condition on the minimum degree required in rn-vertex graphs $G_1, \ldots, G_{\binom{r}{2}n}$ to guarantee a rainbow K_r-factor.

In some of these results, the sufficient condition on each of $G_1, \ldots, G_s$ is the same as that required in a single graph to guarantee a copy of H; in others it is stronger.

As mentioned, there are many open problems in this area and I will try to highlight several of these. Time permitting, I will discuss some even more recent work in this area, including joint work with Yangyang Cheng on regularity tools in this setting.

References

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2. F. Joos and J. Kim, *On a rainbow version of Dirac's theorem*, Bulletin of the London Mathematical Society (2020).
3. R. Montgomery, A. Müyesser and Y. Pehova, *Transversal factors and spanning trees*, Advances in Combinatorics (2021).
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Contributed talks

Universality for degenerate graphs

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A graph G is said to be *universal* for a class $\mathcal{H}$ of graphs if for each $H \in \mathcal{H}$, the graph H is a subgraph (not necessarily induced) of G.

Two particularly interesting questions are: what is the threshold p such that the random graph $G(n,p)$ is universal for $\mathcal{H}$, and what is the minimum $e(G)$ over graphs G which are universal for $\mathcal{H}$? Both questions have been studied for $\mathcal{H}$ the class of n-vertex graphs with maximum degree at most Δ. The latter extremal question is quite well understood: there are explicit constructions due to Alon and coauthors (in several papers) which give the correct order of magnitude $O(n^{2-2/\Delta})$.

The former probabilistic question remains open; conjecturally, the answer is the same as the appearance threshold for a $K_{\Delta+1}$-factor in $G(n,p)$, i.e. $\tilde{\Theta}(n^{-2/(\Delta+1)})$ (by the Johansson-Kahn-Vu theorem). In particular, the random graph is not an optimal universal graph.

A natural question of Alon is what happens for $\mathcal{H}$ the class of n-vertex graphs with degeneracy at most D. Here it is trivial to see that the random graph is not an optimal universal graph. However, perhaps surprisingly, a close relative, a random block model, turns out to be almost optimal. In this talk I will explain briefly why the sparsest universal graph has $\tilde{\Theta}(n^{2-1/D})$ edges.

This is joint work with Julia Böttcher and Anita Liebenau.

Row-Hamiltonian Latin squares and perfect 1-factorisations

Jack Allsop (Monash University)

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A Latin square of order n is an $n \times n$ matrix of n symbols, such that each symbol occurs exactly once in each row and column. Let L be a Latin square of order n. Each pair of distinct rows of L forms a 2-line permutation. If this permutation is a single n-cycle, for any choice of rows, then L is called row-Hamiltonian. Each

Latin square has six conjugate Latin squares, obtained by uniformly permuting the coordinates of its $(\text{row}, \text{column}, \text{symbol})$ triples. Let $\nu(L)$ denote the number of conjugates of L which are row-Hamiltonian. It is known that $\nu(L) \in \{0, 2, 4, 6\}$ and for each $m \in \{0, 2, 6\}$ there are known infinite families of Latin squares with $\nu = m$. We construct the first known infinite family of Latin squares with $\nu = 4$.

A 1-factorisation of a graph is a partition of its edge set into 1-factors. A 1-factorisation is perfect if the union of edges in any pair of its 1-factors forms a Hamiltonian cycle. A perfect 1-factorisation of the complete bipartite graph $K_{n,n}$ is equivalent to a row-Hamiltonian Latin square of order n. Our family of Latin squares with $\nu = 4$ allows us to build the eighth known infinite family of perfect 1-factorisations of complete bipartite graphs. This is joint work with Ian Wanless.

Effective bounds for induced size-Ramsey numbers of cycles

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The induced size-Ramsey number $\hat{r}^k_{\text{ind}}(H)$ of a graph H is the smallest number of edges a (host) graph G can have such that for any k-coloring of its edges, there exists a monochromatic copy of H which is an induced subgraph of G. In 1995, in their seminal paper, Haxell, Kohayakawa and Łuczak showed that for cycles, these numbers are linear for any constant number of colours, i.e., $\hat{r}^k_{\text{ind}}(C_n) \le Cn$ for some $C = C(k)$. The constant C comes from the use of the regularity lemma, and has a tower type dependence on k. In this paper we significantly improve these bounds, showing that $\hat{r}^k_{\text{ind}}(C_n) \le O(k^{102})n$ when n is even, thus obtaining only a polynomial dependence of C on k. We also prove $\hat{r}^k_{\text{ind}}(C_n) \le e^{O(k \log k)}n$ for odd n, which almost matches the lower bound of $e^{\Omega(k)}n$. Finally, we show that the ordinary (non-induced) size-Ramsey number satisfies $\hat{r}^k(C_n) = e^{O(k)}n$ for odd n. This substantially improves the best previous result of $e^{O(k^2)}n$, and is best possible, up to the implied constant in the exponent. To achieve our results, we present a new host graph construction which, roughly speaking, reduces our task to finding a cycle of approximate given length in a graph with local sparsity.

Joint work with Nemanja Draganić and Benny Sudakov.

Large Pure Pairs in Edge-Coloured Graphs

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Given a k-edge-coloured complete graph K_n, a *pure pair* (A, B) is a pair of disjoint vertex sets such that the induced graph $K_n[A, B]$ uses at most $k-1$-colours. Note that in 2-coloured case, a pure pair is a monochromatic complete bipartite subgraph. A conjecture of Conlon, Fox and Sudakov asserts that for every 2-edge-coloured complete graph H, there exists $\varepsilon > 0$ such that every 2-edge-coloured complete graph K_n does not contain a subgraph isomorphic to H has a pure pair (A, B) with $|A| \geq \varepsilon n^{\varepsilon}$ and $|B| \geq \varepsilon n$.

In this talk, I will survey some progress to this conjecture as well as its connection to the famous Erdös-Hajnal conjecture. I will also introduce a generalization of it and some progress we made on it. Joint work with Peter Keevash.

Block avoiding sequencings of Steiner systems

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Abstract: A *partial* (n, k, t)*-Steiner system* is a pair $(X, \mathcal{B})$ where X is an n-set of *vertices* and $\mathcal{B}$ is a collection of k-subsets of X called *blocks* such that each t-set of vertices is a subset of at most one block. An ℓ*-block avoiding sequencing* of such a system is a labelling of its vertices with distinct elements of $\{0, \ldots, n-1\}$ such that no block is contained in a set of ℓ vertices with consecutive labels. This talk will discuss block avoiding point sequencings of partial Steiner systems. In particular, we outline a proof that, for fixed k and t, any partial (n, k, t)-Steiner system has an ℓ-good sequencing for some $\ell = \Theta(n^{1/t})$ as n becomes large. This result is perhaps of most interest in the case $k = t+1$ where results of Kostochka, Mubayi and Verstraëte show that the value of ℓ cannot be increased beyond $\Theta((n \log n)^{1/t})$. A special case of this result shows that every partial Steiner triple system (partial $(n, 3, 2)$-Steiner system) has an ℓ-block avoiding sequencing for each positive integer $\ell \leq 0.0908\, n^{1/2}$. This is joint work with Padraig Ó Catháin.

Rainbow spanning trees in uniformly coloured perturbed graphs

Kyriakos Katsamaktsis, joint work with Shoham Letzter and Amedeo Sgueglia (University College London)

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We consider the problem of finding a copy of a rainbow spanning bounded-degree tree in the uniformly edge-coloured randomly perturbed graph. Let G_0 be an n-vertex graph with minimum degree at least δn, and let T be a tree on n vertices with maximum degree at most d, where $\delta \in (0, 1)$ and $d \geq 2$ are constants. We show that there exists $C = C(\delta, d) > 0$ such that, with high probability, if the edges of the union $G_0 \cup G(n, C/n)$ are uniformly coloured with colours in $[n-1]$, then there is a rainbow copy of T. Our result resolves in a strong form a conjecture of Aigner-Horev, Hefetz and Lahiri[1].

References

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Hypergraphs with minimum positive uniform Turán density

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Reiher, Rödl and Schacht[1] showed that the uniform Turán density of every 3-uniform hypergraph is either 0 or at least 1/27, and asked whether there exist 3-uniform hypergraphs with uniform Turán density equal or arbitrarily close to 1/27. We construct 3-uniform hypergraphs with uniform Turán density equal to 1/27. Joint work with Frederik Garbe and Dan Král'.

References

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Perfect hypergraph tilings

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In the perfect hypergraph tiling problem, we are given hypergraphs G and F. The task then consists in covering the vertices of G with pairwise vertex-disjoint copies of F. There are three essentially necessary conditions for the existence perfect tilings, which correspond to barriers in space, divisibility and covering. It is natural to ask for which classes of hypergraphs these conditions are also asymptotically sufficient.

In this talk, I discuss an approach to this question for hypergraph families that are approximately closed under taking typical induced subgraphs of constant order. Among others, this includes families parametrised by minimum degrees and quasirandomness, which have been studied extensively in this setting.

Computational construction of ovoids in projective geometries

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An arc in a finite projective plane of order q is defined to be a set of k points no three of which are colinear. It is known [2] that for q odd k is at most $q+1$ and for q even k is at most $q+2$. All $(q+1)$-arcs are called ovals, while $(q+2)$-arcs are called hyperovals. In finite projective geometries of odd order all ovals have been classified up to an isomorphism ([1], all such ovals are conics). For projective geometries of even order that is not the case and so this problem is one of the central interesets in finite projective geometries of even order.

In this talk we will survey our work with known hyperovals for projective geometries of even order up to $q = 128$. Using these hyperovals it is possible to construct all known ovals, which are then used to construct ovoids in projective space of the same order. This is useful, for instance, because inversive planes of even order can be constructed from the points of an ovoid [2].

There are 11 known infinite classes of hyperovals for projective planes of even order and, in particular, for $q = 128$ this reduces to 8 known nonisomorphic hyperovals. We will present our implementation based on parallel processing for computational construction of examples of ovoids in $\mathrm{PG}(3, 128)$. The aim of this project is to give us more insight on ovals of projective plane and ovoids of projective space of order 128.

References

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Multistage Maker-Breaker games

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We initiate the study of a new variant of the Maker-Breaker positional game, which we call multistage game. Given a hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{F})$ and a bias $b \geq 1$, the $(1 : b)$ multistage Maker-Breaker game on $\mathcal{H}$ is played in several stages as follows. Each stage is played as a usual $(1 : b)$ Maker-Breaker game, until all the elements of the board get claimed by one of the players, with the first stage being played on $\mathcal{H}$.

In every subsequent stage, the game is played on the board reduced to the elements that Maker claimed in the previous stage, and with the winning sets reduced to those fully contained in the new board. The game proceeds until no winning sets remain, and the goal of Maker is to prolong the duration of the game for as many stages as possible.

Here we estimate the maximum duration of the $(1 : b)$ multistage Maker-Breaker game, for biases b subpolynomial in n, for some standard graph games played on the edge set of K_n: the connectivity game, the Hamilton cycle game, the non-k-colorability game, the pancyclicity game and the H-game. While the first three games exhibit a probabilistic intuition, it turns out that the last two games fail to do so. We give more detail on that, and some open problems related to this topic.

This is joint work with Juri Barkey, Dennis Clemens, Fabian Hamann and Amedeo Sgueglia.

Tranversals in quasirandom latin squares

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A transversal in a $n \times n$ latin square is a set of n entries not repeating any row, column, or symbol. A famous conjecture of Brualdi, Ryser, and Stein predicts that every latin square has at least one transversal provided n is odd. We will discuss an approach motivated by the circle method from the analytic number theory which enables us to count transversals in latin squares which are quasirandom in an appropriate sense.

Size-Ramsey numbers of structurally sparse graphs

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The size-Ramsey number of a graph H is the minimum number of edges of a host graph G such that any k-edge-colouring of G contains a monochromatic copy of H. Research has mainly focused on the size-Ramsey numbers of n-vertex graphs with constant maximum degree D. For example, graphs which also have constant treewidth are known to have linear size-Ramsey numbers. On the other extreme, the canonical examples of graphs of unbounded treewidth are the grid graphs, for which the best known bound has only very recently been improved from $O(n^{3/2})$ to $O(n^{5/4})$ by Conlon, Nenadov and Trujic. In this work, we prove a common generalization of these results by establishing new bounds on the size-Ramsey numbers in terms of treewidth (which may grow as a function of n). As a special case, this yields a bound of $\tilde{O}(n^{3/2-1/(2D)})$ for proper minor-closed classes of graphs. In particular, this bound applies to planar graphs, addressing a question of Wood. Our proof combines methods from structural graph theory and classic Ramsey-theorertic embedding techniques, taking advantage of the product structure exhibited by graphs with bounded treewidth. This is joint work with Nemanja Draganic, Marc Kaufmann, David Munha Correia, and Raphael Steiner.

Maker-Breaker games on random boards

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In Maker-Breaker games played on edge sets of graphs, two players, Maker and Breaker, alternately claim unclaimed edges of a given graph until all of its edges are claimed. Maker wins the game if he claims all edges of one representative of a prescribed graph-theoretic structure (e.g. a Hamiltonian cycle, or a fixed graph H). Breaker wins otherwise.

We take a closer look at various Maker-Breaker games played on the edge sets of random graphs, encountering some open problems along the way.

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